Models Rush In Where Simulators Fear To Tread: Extending the Baseband-Equivalent Method

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Abstract

Radio-frequency (RF) receiver circuits have operating characteristics that make them difficult to simulate using traditional SPICE transient analysis. Their gross operation alone, running at gigahertz frequencies, where cycle-by-cycle simulations will overwhelm the fastest computers and the results will rapidly fill disks. Another bottleneck is sifting through that data.

For system design work, these problems can be overcome by using Verilog-A behavioral models to remove the high-frequency carrier signal from the simulation. Based on the same concepts as the baseband-equivalent models (available in Cadence's "rfLib" library) but extended to handle interference, the "spectral model" is based on an algebraic formula derived to model nonlinear amplification of multiple signals.

To handle frequency-dependent response, filtering, and impedance matching, equivalent models for linear devices are used like their baseband-equivalent counterparts to achieve correct results. This approach, wrapping a nonlinear power-series model with linear frequency-dependent components, is an equivalent-circuit alternative to the generally accepted need to use Volterra series for an accurate "black box" nonlinear model.

With a bit of math (and no Verilog-A code), the spectral model for an amplifier is developed and compared with the traditional approach for simulation speed and accuracy in reproducing the effects of blocking, intermodulation, and cross modulation.

Baseband-Equivalent Amplifier Model

The general form of a modulated signal is

$$\begin{aligned} x &= I(t)\cos(\omega_c t) - Q(t)\sin(\omega_c t) \\ &= \sqrt{I^2(t)+Q^2(t)}\,\cos\left(\omega_c + \tan^{-1}\left(Q(t) \div I(t)\right)\right) \\ &= \mathrm{Re}\left(\left(I(t)+\mathbf{j}Q(t)\right)e^{\mathbf{j}\omega_c t}\right), \end{aligned} \tag{1}$$

a real-valued signal, where $I(t)$ and $Q(t)$ are the in-phase and quadrature components of the baseband signal, respectively. Geometrically speaking, this represents a signal that is in a fixed reference frame, with an instantaneous magnitude and phase applied to the carrier frequency ω_c. Likewise, the **baseband-equivalent signal** represents the same signal in a reference frame that "rotates" with the carrier frequency, and has a complex value

$$\begin{aligned} x_b &= I(t)+\mathbf{j}Q(t) \\ &\equiv I+\mathbf{j}Q. \end{aligned} \tag{2}$$

A transfer function of a memory-less nonlinear block, such as an amplifier, can be expressed as a Taylor series

$$y = \sum_{k=0}^{\infty} c_k\, x^k \tag{3}$$

where the second and higher-order terms account for the nonlinearity. Using a trigonometric relation and its special case

$$\begin{aligned} \cos(\alpha)\cos(\beta) &= \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2} \\ \cos(\alpha)^2 &= \frac{1+\cos(2\alpha)}{2}, \end{aligned} \tag{4}$$

a narrowband input to this block

$$\begin{aligned} x &= A(t)\cos(\omega_0 t) \\ &\equiv A\cos(\omega_0 t) \end{aligned} \tag{5}$$

provides a narrowband output at the fundamental, amidst the binomial pattern of harmonics generated, found by summing the same frequency terms and ignoring all harmonic frequencies (which would be filtered out by bandpass filters and the limited frequency response of the circuit)

$$\begin{aligned} y\big|_{\omega_0} &= \cos(\omega_0 t)\left[c_1 A + 3\frac{c_3}{4}A^3 + 10\frac{c_5}{16}A^5 + \cdots\right] \\ &= \cos(\omega_0 t)\sum_{k=0}^{\infty}\frac{c_{2k+1}}{2^{2k}}\binom{2k+1}{k}A^{2k+1} \end{aligned} \tag{6}$$

and recalling the definition of a binomial coefficient, sometimes written $(n;k)$

$$\begin{aligned} \binom{n}{k} = \binom{n}{n-k} &\equiv \frac{n!}{k!\,(n-k)!} \\ &= \frac{n}{1}\frac{n-1}{2}\cdots\frac{n-k+1}{k} \\ &= \frac{n}{1}\frac{n-1}{2}\cdots\frac{k+1}{n-k} \quad \text{where } 0 \le k \le n, \end{aligned} \tag{7}$$

which is the number of ways of choosing k objects from a collection of n distinct objects without regard to order. Note that in (6), with only one modulated carrier, only odd-ordered terms contribute to nonlinearity at the fundamental frequency.

Repeating the development of (6) using a baseband representation gives

$$y_b = (I + \mathbf{j}Q)\begin{bmatrix} c_1 + 3\frac{c_3}{4}(I^2+Q^2) \\ +10\frac{c_5}{16}(I^2+Q^2)^2 + \cdots \end{bmatrix} \\ = x_b \sum_{k=0}^{\infty} \frac{c_{2k+1}}{2^{2k}}\binom{2k+1}{k}\left|x_b\right|^{2k} . \tag{8}$$

Baseband modeling, in effect, discards the cosine factor from (6) and uses only the remaining series for the model (and of that, only the very first, few terms are needed for accuracy). Note how the complex part of this result is contained *entirely in the leading term* and is a factor for the series, which simplifies extracting the modulation terms to

$$\begin{aligned} I(t) &= \ \operatorname{Re}(y_b) = \ \operatorname{Re}(x_b)\sigma \\ Q(t) &= -\operatorname{Im}(y_b) = -\operatorname{Im}(x_b)\sigma \end{aligned} \tag{9}$$

where

$$\sigma = \sum_{k=0}^{\infty} \frac{c_{2k+1}}{2^{2k}}\binom{2k+1}{k}\left|x_b\right|^{2k} . \tag{10}$$

Summary: In this section, we developed a Taylor-series approximation of the distortion imposed by amplifier nonlinearity on the in-phase and quadrature components of a modulated signal with the carrier suppressed. This is the basis of the **baseband-equivalent model** for the amplifier.

Baseband-Equivalent Models for Common Devices

Even in a system-level design, it is convenient to employ a simple device like the lowly resistor. A two-terminal device normally, the resistor would need to transform into a four-terminal device (two pairs of terminals) to account for the in-phase and quadrature components of the voltage across and current through the device, which we derive as

$$\begin{aligned} \mathit{current} &= V \div R \\ &= \left(I(t)\cos(\omega_c t) - Q(t)\sin(\omega_c t)\right) \div R \\ &= \frac{I(t)}{R}\cos(\omega_c t) - \frac{Q(t)}{R}\sin(\omega_c t) \\ &= \frac{\sqrt{I^2(t)+Q^2(t)}}{R}\cos\left(\omega_c + \tan^{-1}\left(\frac{Q(t)}{I(t)}\right)\right) \\ &= \operatorname{Re}\left(\frac{I(t)+\mathbf{j}Q(t)}{R}e^{\mathbf{j}\omega_c t}\right). \end{aligned} \tag{11}$$

To find the baseband-equivalent signals, we simply write the traditional equations in exponential form, eliminate the exponential, then separate the real and imaginary parts, for example

$$\begin{aligned} V &= I \cdot R \\ &\Downarrow \\ (V_I+\mathbf{j}V_Q)\,e^{\mathbf{j}\omega t} &= (I_I+\mathbf{j}\,I_Q)\,e^{\mathbf{j}\omega t}R \\ &\Downarrow \\ V_I+\mathbf{j}V_Q &= (I_I+\mathbf{j}\,I_Q)R \\ &\Downarrow \\ V_I &= I_I R \quad \text{in-phase "}I\text{-signal"} \\ V_Q &= I_Q R \quad \text{quadrature "}Q\text{-signal"} \end{aligned} \tag{12}$$

This suggests that, beyond the mathematical convenience of working with I and Q as factors of the actual signal, we can also treat them as *real signals in the simulator* to calculate the components of the resulting current. As shown in Fig. 1, the "black box" of the baseband-equivalent four-terminal resistor is simply a pair of identical resistors, one for each the I-signal path and Q-signal path.

The step eliminating the exponential is less trivial for a reactive component. With an inductor, for example we have to differentiate using the chain rule and collect terms before canceling the exponential.

$$\begin{aligned} V &= L\tfrac{d}{dt}I \\ &\Downarrow \\ (V_I+\mathbf{j}V_Q)\,e^{\mathbf{j}\omega t} &= L\tfrac{d}{dt}\left((I_I+\mathbf{j}\,I_Q)\,e^{\mathbf{j}\omega t}\right) \\ &= L\left(\tfrac{d}{dt}(I_I+\mathbf{j}\,I_Q) + \mathbf{j}\omega\,(I_I+\mathbf{j}\,I_Q)\right)e^{\mathbf{j}\omega t} \\ &= L\left(\tfrac{d}{dt}(I_I+\mathbf{j}\,I_Q) + \omega\,(\mathbf{j}\,I_I-I_Q)\right)e^{\mathbf{j}\omega t} \\ &\Downarrow \\ V_I+\mathbf{j}V_Q &= L\left(\tfrac{d}{dt}(I_I+\mathbf{j}\,I_Q) + \omega\,(\mathbf{j}\,I_I-I_Q)\right) \\ &\Downarrow \\ V_I &= L\left(\tfrac{d}{dt}I_I - \omega\,I_Q\right) \\ V_Q &= L\left(\tfrac{d}{dt}I_Q + \omega\,I_I\right) \end{aligned} \tag{13}$$

As shown in Fig. 2, inside the "black box" of the baseband-equivalent four-terminal inductor is a pair of identical inductors (no mutual coupling), each in series with a current-controlled voltage source (transresistance) controlled by the

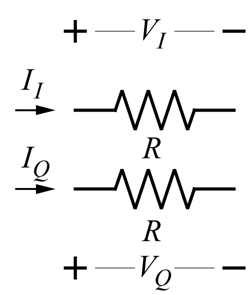

Fig. 1. Baseband-equivalent resistor.

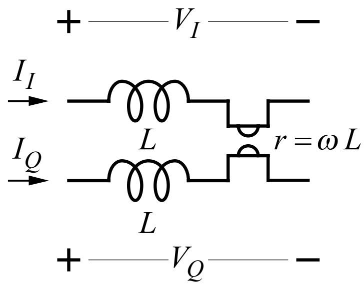

Fig. 2. Baseband-equivalent inductor.

current through the opposite branch. These cross-coupled transresistances form a **gyrator**[1], with a gyration resistance of ωL.

Finding a gyrator in the equivalent circuit model for an inductor is a consequence of making the abstraction of I and Q as electrical signals. The steps we took to remove the carrier are equivalent to the geometric transformations Park developed in the 1920s for modeling motors, converting from a fixed (passband) to a rotating (baseband) frame of reference. An inductor is the mechanical equivalent of mass or inertia, specifically the rotational inertia of a motor's rotor. The gyrator in the equivalent circuit corresponds to the "speed voltage" found in these motor models and the "back EMF" in spinning motors.

Likewise, for a capacitor we have to differentiate using the chain rule and collect terms before canceling the exponential.

$$
\begin{aligned}
I &= C\tfrac{d}{dt}V \\
&\Downarrow \\
(I_I+\mathbf{j}I_Q)\,e^{\mathbf{j}\omega t} &= C\tfrac{d}{dt}\Big((V_I+\mathbf{j}V_Q)\,e^{\mathbf{j}\omega t}\Big) \\
&= C\Big(\tfrac{d}{dt}(V_I+\mathbf{j}V_Q)+\mathbf{j}\omega\,(V_I+\mathbf{j}V_Q)\Big)e^{\mathbf{j}\omega t} \\
&= C\Big(\tfrac{d}{dt}(V_I+\mathbf{j}V_Q)+\omega\,(\mathbf{j}V_I-V_Q)\Big)e^{\mathbf{j}\omega t} \\
&\Downarrow \\
I_I+\mathbf{j}I_Q &= C\Big(\tfrac{d}{dt}(V_I+\mathbf{j}V_Q)+\omega\,(\mathbf{j}V_I-V_Q)\Big) \\
&\Downarrow \\
I_I &= C\Big(\tfrac{d}{dt}V_I-\omega V_Q\Big) \\
I_Q &= C\Big(\tfrac{d}{dt}V_Q+\omega V_I\Big)
\end{aligned}
\tag{14}
$$

As shown in Fig. 3, inside the "black box" of the baseband-equivalent four-terminal inductor is a pair of identical capacitors, each in parallel with a voltage-controlled current source (transconductance) controlled by the voltage across the opposite branch. These cross-coupled transconductances form

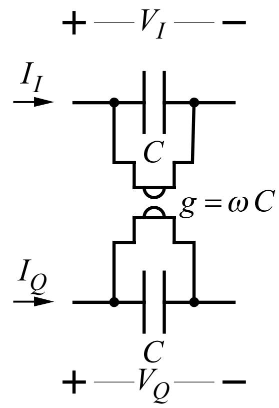

Fig. 3. Baseband-equivalent capacitor.

(you guessed it) a gyrator, with a gyration conductance of ωC.

From linear system theory, we know that for linear circuits each frequency is independent of any other frequency. So each I/Q signal pair is a separate circuit operating at its own assumed frequency and there is no connection or coupling between these signals. If the devices are nonlinear, only then is there coupling between frequencies.

Summary: In this section, we developed the notion of treating I and Q as real electrical signals. This is an abstraction within an abstraction: (a) the baseband-equivalent signal is an abstraction of the actual signal, and (b) handling these baseband signals as though they are "real" voltages is an abstraction for employing the simulator, via Kirchhoff's laws, to calculate baseband currents.

Spectral-Equivalent Amplifier Model

Returning to our Taylor series representation (3) of an amplifier and increasing the number of inputs

$$
\begin{aligned}
x &= x_1 + x_2 + \cdots + x_M \\
&= \sum_{m=1}^{M} x_m\ ,
\end{aligned}
\tag{15}
$$

where

$$x_m = I_m \cos(\omega_m t) - Q_m \sin(\omega_m t)\ , \tag{16}$$

then (3) can be written as

$$y = \sum_{k=0}^{\infty} c_k \left(\sum_{m=1}^{M} x_m\right)^k . \tag{17}$$

The right-most factor for each term on the right-hand side is a multinomial expansion, which can be written as

$$\left(\sum_{m=1}^{M} x_m\right)^k = \underbrace{\sum_{n_1}\sum_{n_2}\cdots\sum_{n_M}}_{n_1+n_2+\cdots+n_M=k} \binom{k}{n_1, n_2, \ldots, n_M} \prod_{m=1}^{M} x_m^{n_m}\ , \tag{18}$$

where the summation on the right-hand side is over all sets of

[1] A device postulated by Tellegen in 1948. He called it a gyrator, referring to a mechanical analogy with spinning flywheels (gyroscopes) such that it "gyrates a current into voltage and vice versa."

non-negative integers (the *natural numbers* 0, 1, …, ∞) that sum to k, which is a "short hand" equivalent to

$$\underbrace{\sum_{n_1}\sum_{n_2}\cdots\sum_{n_M}}_{n_1+n_2+\cdots+n_M=k} \equiv \sum_{n_1=0}^{k}\sum_{n_2=0}^{k-n_1}\cdots\sum_{n_{M-1}=0}^{k-n_1-\cdots-n_{M-2}}\;\sum_{n_M=k-n_1-\cdots-n_{M-1}} . \qquad (19)$$

Also, recall the definition of a multinomial coefficient is defined as

$$\begin{aligned}\binom{k}{n_1, n_2, \ldots, n_M} &\equiv \frac{k!}{n_1!\, n_2! \cdots n_M!} \text{ where } k = \sum_{m=1}^{M} n_m \\ &\equiv \frac{(n_1+n_2+\cdots+n_M)!}{n_1!\, n_2! \cdots n_M!}\,,\end{aligned} \qquad (20)$$

which is the number of ways of putting k distinct objects into M different boxes with n_m objects in the m^{th} box. When $M = 2$, it becomes the binomial coefficient.

The product in the summation on the right-hand side of (18) can be rewritten using Euler's identities and complex conjugates as

$$\begin{aligned}\prod_{m=1}^{M} x_m^{n_m} &= \prod_{m=1}^{M}\left(I_m\cos(\omega_m t) - Q_m\sin(\omega_m t)\right)^{n_m} \\ &= \prod_{m=1}^{M}\left(\frac{(I_m+\mathbf{j}Q_m)\,e^{\mathbf{j}\omega_m t} + (I_m-\mathbf{j}Q_m)\,e^{-\mathbf{j}\omega_m t}}{2}\right)^{n_m} \\ &= \prod_{m=1}^{M}\left(\frac{A_m\, e^{\mathbf{j}\omega_m t} + \overline{A}_m\, e^{-\mathbf{j}\omega_m t}}{2}\right)^{n_m}\end{aligned} \qquad (21)$$

where $A_m = I_m + \mathbf{j}Q_m$, $\overline{A}_m = I_m - \mathbf{j}Q_m$. This can be rewritten using a multinomial expansion, rearranged to enumerate the sums of the products, and applying the multinomial condition (20) gives

$$\begin{aligned}\prod_{m=1}^{M} x_m^{n_m} &= \prod_{m=1}^{M}\frac{1}{2^{n_m}}\sum_{k_m=0}^{n_m}\binom{n_m}{k_m} \\ &\quad\cdot\left(A_m e^{\mathbf{j}\omega_m t}\right)^{k_m}\left(\overline{A}_m e^{-\mathbf{j}\omega_m t}\right)^{n_m-k_m} \\ &= \prod_{m=1}^{M}\frac{1}{2^{n_m}}\sum_{k_m=0}^{n_m}\frac{n_m}{k_m!(n_m-k_m)!}A_m^{k_m}\,\overline{A}_m^{\,n_m-k_m} \\ &\quad\cdot e^{\mathbf{j}(2k_m-n_m)\omega_m t} \\ &= \frac{1}{2^k}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\cdots\sum_{k_M=0}^{n_M}\left(\prod_{p=1}^{M}\frac{n_p A_p^{k_p}\,\overline{A}_p^{\,n_p-k_p}}{k_p!(n_p-k_p)!}\right) \\ &\quad\cdot e^{\mathbf{j}\sum_{m=1}^{M}(2k_m-n_m)\omega_m t}\,.\end{aligned} \qquad (22)$$

We can now write (18) in expanded form as

$$\begin{aligned}\left(\sum_{m=1}^{M} x_m\right)^k &= \underbrace{\sum_{n_1}\sum_{n_2}\cdots\sum_{n_M}}_{n_1+n_2+\cdots+n_M=k}\frac{k!}{2^k}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\cdots\sum_{k_M=0}^{n_M} \\ &\quad\left(\prod_{p=1}^{M}\frac{A_p^{k_p}\,\overline{A}_p^{\,n_p-k_p}}{k_p!(n_p-k_p)!}\right)e^{\mathbf{j}\sum_{m=1}^{M}(2k_m-n_m)\omega_m t}\,,\end{aligned} \qquad (23)$$

which involves frequencies of the form

$$\omega = \alpha_1\omega_1 + \alpha_2\omega_2 + \cdots + \alpha_M\omega_M \qquad (24)$$

where each α is any integer.

The right-hand-side of (23) is the full output of all frequency components from the amplifier, whereas we will actually want to calculate the output at a particular frequency. Then the part that remains when a particular frequency is chosen is

$$\begin{aligned}x_k^* &= x_k\left(\cos(\alpha_1\omega_1 t + \alpha_2\omega_2 t + \cdots + \alpha_M\omega_M t)\right. \\ &\qquad\left.+\,\mathbf{j}\sin(\alpha_1\omega_1 t + \alpha_2\omega_2 t + \cdots + \alpha_M\omega_M t)\right) \\ &= \mathrm{Re}(x_k)\cos(\alpha_1\omega_1 t + \alpha_2\omega_2 t + \cdots + \alpha_M\omega_M t) \\ &\qquad - \mathrm{Im}(x_k)\sin(\alpha_1\omega_1 t + \alpha_2\omega_2 t + \cdots + \alpha_M\omega_M t)\end{aligned} \qquad (25)$$

from which we extract

$$\begin{aligned}I_k &= \ \ \mathrm{Re}(x_k) \\ Q_k &= -\mathrm{Im}(x_k)\,.\end{aligned} \qquad (26)$$

Now it is time to discard irrelevant terms and simplify the equations. The only terms that can contribute to a chosen frequency are those for which either

$$\left.\begin{aligned}2k_m - n_m &= +\alpha_m \\ 2k_m - n_m &= -\alpha_m\end{aligned}\right\}\text{ for }\underline{\text{all}}\ m = 1, 2, \ldots, M. \qquad (27)$$

When these two conditions for k_m are used

$$x_k = \underbrace{\sum_{n_1}\sum_{n_2}\cdots\sum_{n_M}}_{n_1+n_2+\cdots+n_M=k}\frac{k!}{2^k}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\cdots\sum_{k_M=0}^{n_M}\begin{cases}\displaystyle\prod_{p=1}^{M}\frac{A_p^{\frac{n_p+\alpha_p}{2}}\,\overline{A}_p^{\,\frac{n_p-\alpha_p}{2}}}{\left(\frac{n_p+\alpha_p}{2}\right)!\left(\frac{n_p-\alpha_p}{2}\right)!} & \boxed{\text{If } 2k_p-n_p=+\alpha_p \text{ for } \underline{\text{all}}\ p\text{, or…}} \\ \displaystyle\prod_{p=1}^{M}\frac{\overline{A_p^{\frac{n_p-\alpha_p}{2}}\,\overline{A}_p^{\,\frac{n_p+\alpha_p}{2}}}}{\left(\frac{n_p-\alpha_p}{2}\right)!\left(\frac{n_p+\alpha_p}{2}\right)!} & \boxed{\text{if } 2k_p-n_p=-\alpha_p \text{ for } \underline{\text{all}}\ p\text{,…}} \\ 0 & \boxed{\text{otherwise.}}\end{cases} \qquad (28)$$

This looks more like an algorithm than an equation. Obviously, you could peak ahead to see if the "otherwise" condition would happen to avoid wasted calculations. Also, if the frequency chosen is DC, use either branch since then both conditions are equivalent—in other words, the term only counts once, not twice.

To account for negated frequencies, the $-\alpha$ branch calculation takes the complex conjugate of its terms. This is equivalent to the $+\alpha$ branch calculation, so the equation condenses to

$$x_k = \underbrace{\sum_{n_1}\sum_{n_2}\cdots\sum_{n_M}}_{n_1+n_2+\cdots+n_M=k} \frac{k!}{2^k}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\cdots\sum_{k_M=0}^{n_M} \begin{cases} \prod_{p=1}^{M} \dfrac{A_p^{\frac{n_p+\alpha_p}{2}}\,\overline{A}_p^{\frac{n_p-\alpha_p}{2}}}{\left(\dfrac{n_p+\alpha_p}{2}\right)!\left(\dfrac{n_p-\alpha_p}{2}\right)!} & \boxed{\begin{array}{c}\text{If } 2k_p-n_p=+\alpha_p \\ \text{for } \underline{\text{all}}\ p\text{, or} \\ \text{if } 2k_p-n_p=-\alpha_p \\ \text{for } \underline{\text{all}}\ p,\ldots\end{array}} \\ 0 & \boxed{\text{otherwise.}} \end{cases} \tag{29}$$

With this equation the DC case is handled correctly (the branch is only used once).

Having accomplished the calculation by rejecting unwanted frequencies, it would be efficient to eliminate the "otherwise" condition. This is possible by observing that the *only* terms that can *really* contribute to the chosen intermodulation are

$$n_m - 2q_m = |\alpha_m| \quad \text{where } q_m = 0, 1, 2, \ldots \tag{30}$$

Since the **order** of an intermodulation frequency is defined by

$$N = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_M| \tag{31}$$

the multinomial-summation condition can be narrowed further to

$$q_1 + q_2 + \cdots + q_M = \tfrac{1}{2}(k-N)\,. \tag{32}$$

Consequently, the output at a particular order of intermodulation frequency is reduced to

$$x_k = \underbrace{\sum_{q_1}\sum_{q_2}\cdots\sum_{q_M}}_{q_1+q_2+\cdots+q_M=\frac{1}{2}(k-N)} \frac{k!}{2^{k-1}}\prod_{p=1}^{M} \begin{cases} \dfrac{A_p^{q_p+\alpha_p}\,\overline{A}_p^{q_p}}{(q_p+\alpha_p)!\,q_p!} & \boxed{\text{if } \alpha_p \ge 0} \\ \dfrac{A_p^{q_p}\,\overline{A}_p^{q_p-\alpha_p}}{q_p!\,(q_p-\alpha_p)!} & \boxed{\text{if } \alpha_p < 0} \end{cases} \tag{33}$$

With further work, we eliminate the branch in the product by letting the absolute value function perform this implicitly

$$x_k = \underbrace{\sum_{q_1}\sum_{q_2}\cdots\sum_{q_M}}_{q_1+q_2+\cdots+q_M=\frac{1}{2}(k-N)} \frac{k!}{2^{k-1}}\prod_{p=1}^{M} \frac{A_p^{q_p+\frac{1}{2}(|\alpha_p|+\alpha_p)}\,\overline{A}_p^{q_p+\frac{1}{2}(|\alpha_p|-\alpha_p)}}{(q_p+|\alpha_p|)!\,q_p!}\,. \tag{34}$$

This is precisely the result given by Sea for amplitude-modulated signals (without phase, these signals equal their own complex conjugate), which simplifies the product terms

$$\underbrace{A_p^{2q_p+|\alpha_p|}}_{\text{Sea's result...}} = \underbrace{A_p^{q_p+\frac{1}{2}(|\alpha_p|+\alpha_p)}\,\overline{A}_p^{q_p+\frac{1}{2}(|\alpha_p|-\alpha_p)}}_{\text{...including } phase!} \tag{35}$$

when $\mathrm{Im}(A_p) = 0$, so that $A_p = \overline{A}_p$.

Inspecting the difference with Sea's result, we see that the magnitude of components for each intermodulation product remains the same

$$\begin{aligned} \left|A_p^{q_p+\frac{1}{2}(|\alpha_p|+\alpha_p)}\,\overline{A}_p^{q_p+\frac{1}{2}(|\alpha_p|-\alpha_p)}\right| &= \left|A_p^{2q_p+|\alpha_p|}\right| \\ &= |A_p|^{2q_p+|\alpha_p|}\,, \end{aligned} \tag{36}$$

while the phase of these components, which was <u>not</u> part of Sea's work, is simply

$$\begin{aligned} \angle\left(A_p^{q_p+\frac{1}{2}(|\alpha_p|+\alpha_p)}\,\overline{A}_p^{q_p+\frac{1}{2}(|\alpha_p|-\alpha_p)}\right) &= \angle A_p^{\alpha_p} \\ &= \alpha_p\,\angle A_p\,. \end{aligned} \tag{37}$$

These factors describe the dreaded **cross modulation** of signals.

We can improve (34) by eliminating the factorials, which increase exponentially presenting difficulties in calculation, by combining terms to form a multinomial coefficient

$$\begin{aligned} k!\prod_{p=1}^{M}\frac{1}{(q_p+|\alpha_p|)!\,q_p!} &= \frac{k!}{\prod_{p=1}^{M}(q_p+|\alpha_p|)!\,q_p!} \\ &= \binom{k}{q_1,\, q_1+|\alpha_1|,\ldots,\, q_M,\, q_M+|\alpha_M|} \\ &\text{where } k = 2\sum_{p=1}^{M} q_p + \sum_{p=1}^{M}|\alpha_p|\,, \end{aligned} \tag{38}$$

which represents the multiplier on signal combinations creating an intermodulation (IM) product. Then we can rewrite the product to separate that which is required in the product from that which is constant over the multi-summation

$$
\begin{aligned}
&A_p^{q_p+\frac{1}{2}(|\alpha_p|+\alpha_p)}\,\overline{A}_p^{\,q_p+\frac{1}{2}(|\alpha_p|-\alpha_p)} \\
&= |A_p|^{2q_p}\cdot\begin{cases} A_p^{\alpha_p} & \boxed{\text{if } \alpha_p>0} \\ 1 & \boxed{\text{if } \alpha_p=0} \\ \overline{A}_p^{\,-\alpha_p} & \boxed{\text{if } \alpha_p<0} \end{cases} \\
&= (I_p^2+Q_p^2)^{q_p}\cdot\begin{cases} (I_p+\mathbf{j}Q_p)^{\alpha_p} & \boxed{\text{if } \alpha_p>0} \\ 1 & \boxed{\text{if } \alpha_p=0} \\ (I_p-\mathbf{j}Q_p)^{-\alpha_p} & \boxed{\text{if } \alpha_p<0} \end{cases}
\end{aligned}
\tag{39}
$$

that can be move out to the front of the formula. These changes produce a formula that elegantly represents how IM products are produced. Picking up from (34)

$$
\begin{aligned}
x_k &= \underbrace{\sum_{q_1}\sum_{q_2}\cdots\sum_{q_M}}_{q_1+q_2+\cdots+q_M=\frac{1}{2}(k-N)} \frac{k!}{2^{k-1}}\prod_{p=1}^{M} \frac{A_p^{q_p+\frac{1}{2}(|\alpha_p|+\alpha_p)}\,\overline{A}_p^{\,q_p+\frac{1}{2}(|\alpha_p|-\alpha_p)}}{(q_p+|\alpha_p|)!\,q_p!} \\
&= \frac{\beta}{2^{k-1}} \underbrace{\sum_{q_1}\sum_{q_2}\cdots\sum_{q_M}}_{q_1+q_2+\cdots+q_M=\frac{1}{2}(k-N)} \binom{k}{q_1,\, q_1+|\alpha_1|, \ldots, q_M,\, q_M+|\alpha_M|}\prod_{p=1}^{M}|A_p|^{2q_p}
\end{aligned}
\tag{40}
$$

where

$$
\beta = \prod_{p=1}^{M}\begin{cases} A_p^{\alpha_p} & \boxed{\text{if } \alpha_p>0} \\ 1 & \boxed{\text{if } \alpha_p=0} \\ \overline{A}_p^{\,-\alpha_p} & \boxed{\text{if } \alpha_p<0} \end{cases} \tag{41}
$$

Looking at the multinomial-summation condition (32), we can see that the result will be zero if $k-N$ is not an even, non-negative integer. This fact represents our "radio knowledge" that an intermodulation of order N can only be produced by terms in the Taylor series (3) of degree k greater than or equal to N, <u>and</u> that k must be even or odd as N is even or odd. Combining this fact with (3) we find the total intermodulation amplitude to be

$$
y_N = c_N\, x_N + c_{N+2}\, x_{N+2} + c_{N+4}\, x_{N+4} + \cdots, \tag{42}
$$

which provides our "final answer" of

$$
y_N = \beta\sum_{L=0}^{\infty}\frac{c_{2L+N}}{2^{2L+N-1}} \underbrace{\sum_{q_1}\sum_{q_2}\cdots\sum_{q_M}}_{q_1+q_2+\cdots+q_M=L} \binom{2L+N}{q_1,\, q_1+|\alpha_1|, \ldots, q_M,\, q_M+|\alpha_M|}\prod_{p=1}^{M}|A_p|^{2q_p} \tag{43}
$$

Note how the complex part of the answer is contained *entirely in the leading term* and is a factor for the series, which simplifies extracting the modulation terms to

$$
\begin{aligned}
I(t) &= \operatorname{Re}(y_N) = \operatorname{Re}(\beta)\,\sigma \\
Q(t) &= -\operatorname{Im}(y_N) = -\operatorname{Im}(\beta)\,\sigma
\end{aligned}
\tag{44}
$$

where

$$
\sigma = \sum_{L=0}^{\infty}\frac{c_{2L+N}}{2^{2L+N-1}} \underbrace{\sum_{q_1}\sum_{q_2}\cdots\sum_{q_M}}_{q_1+q_2+\cdots+q_M=L} \binom{2L+N}{q_1,\, q_1+|\alpha_1|, \ldots, q_M,\, q_M+|\alpha_M|}\prod_{p=1}^{M}|A_p|^{2q_p} \tag{45}
$$

As discussed above, the DC output is 1/2 of these terms and, since $N=0$, (43) simplifies to

$$
y_N = \sum_{L=0}^{\infty}\frac{c_{2L}}{2^{2L}} \underbrace{\sum_{q_1}\sum_{q_2}\cdots\sum_{q_M}}_{q_1+q_2+\cdots+q_M=L} \binom{2L}{q_1, q_1, \ldots, q_M, q_M}\prod_{p=1}^{M}|A_p|^{2q_p}. \tag{46}
$$

This is precisely the result given by Sea (each signal cancels its own phase). Another sanity check, (44) and (45) with $M=1$ simplify to (9) and (10) respectively.

Summary: In this section, we have taken the baseband model approach and expanded it to include any number of signals, assuming each is a modulated carrier. This is the basis for the spectral-equivalent model of the amplifier. Because the amplifier is nonlinear, the solution shows how cross modulation occurs at any output "spectral" frequency.

A Small Benchmark

Assume three CDMA modulated signals at 2GHz, 2.01GHz, and 2.02GHz, the desired signal being at 2GHz and the others are interference. This setup provides the classic example of worst case IM3 interference and cross modulation.

This schematic in Fig. 4 is a "master" from which parts were extracted to perform circuit simulation and obtain run times for evaluating the speed-up from using the spectral model. On the left side are three CDMA generators, modified from one in the Cadence library, that generates $I(t)$ and $Q(t)$ for random data. Each generator uses a different seed for the pattern, to create a different sequence, but the sequences remain in lock step (that is, the chips transition at simultaneously). Then ideal multipliers serve as mixers to create the carriers, which are summed together. The signal level of the desired signal is set to 10mV peak (–30dBm) and the interference signals are both set to 100mV peak (–10dBm). The both LNA models are set to a voltage gain of 5 (13.98dB) into 50Ω and the IIP_3 is set to 6dBm. For the passband LNA, which is a Verilog-A model from the Cadence library, a pair of ideal multipliers mix-down the signal, which is then heavily filtered by a 10-pole, low-pass Bessel filter "brick wall" with its –3dB point set at the CDMA chip rate of 1.2288MHz to eliminate the unwanted signals from 10MHz and 20MHz (and various 6GHz harmonics mixed to 4GHz and 8GHz). The filter model contains one line of Verilog-A, albeit highly parameterized with pole locations. Because the group delay of this filter is noticeable, the reference signals and outputs from the spectral model have similar filters.

The waveforms in Fig. 5 show the CDMA signals. Note the scale change for the leftmost plot. There is roughly a 4µs delay in CDMA output as the model includes a 48-stage digital filter to band limit the signals.

The waveforms in Fig. 6 compare the distorted and ideal $I(t)$ and $Q(t)$. Cross modulation is seriously degrading these signals.

The waveforms in Fig. 7 compare the "old" passband model and "new" spectral model results. On the left, agreement looks good. On the right, the differences in the waveforms are magnified to show about 1% difference. This turned out to be an artifact of the "brick wall" filters, which were leaking the envelope of spurious signals offset at 10MHz and 20MHz from the passband LNA. In a separate simulation, the modulated 2.01GHz and 2.20GHz signals were individually amplified by identical passband LNAs and these outputs combined before down mixing by 2.00GHz and filtering. The output was identical to the difference shown above. This demonstrates that the spectral model is providing the truly correct response!

The simulation statistics in Table 1 reveal that the number of time points evaluated was determined by the carrier oscillators, for the "old" passband model, but seems to be set by the CDMA source for the "new" spectral model. Even the reference circuit, with only the "brick wall" filters, used nearly as many points. Another set of runs for 20µs simulated time with a 10ps step limit (about 2 million time points) required 6,235 seconds for the "new" spectral model, but only 1,992 seconds for the "old" passband model—a third the performance in simulation efficiency per time point. By simulating at the chip rate of 1.3MHz instead of the 2GHz carrier rate, the spectral technique gains a factor of 1,500 in reduced workload. A reasonable figure is "500+" for the speedup for using the spectral model.

In addition, the conservative runs in Table 1 produced a 2.4G-byte waveform file for the "old" passband model, and a modest 434k-byte waveform file for the "new" spectral model—a reduction of over 5,000 in file size and speedup in waveform plotting!

Summary: A small benchmark comparing LNA models (nonlinear passband model versus spectral model) showed a 500x simulation speed improvement and a 5,000x reduction in output data. The speed improvement ratio is equivalent to condensing eight hours into one minute.

Conclusion

We have shown how to extend the baseband-equivalent method to model passive devices and explicitly include the nonlinear effects of blocking, intermodulation, and cross modulation. By mathematically suppressing <u>all</u> carriers, the "spectral" model greatly accelerates time-domain simulation.

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Table 1. Simulation results.

200µs simulated	accuracy setting	time points generated	seconds used	time points per second	seconds per µs simulated	"speedup"
"old"	conservative	8,288,000	9,880	839	49.4	
	moderate	4,853,000	6,188	784	30.9	
	liberal	3,626,000	4,591	790	23.0	
"new"	conservative	3,897	20.46	190	0.102	705
	moderate	3,722	19.56	190	0.098	462
	liberal	3,043	15.23	200	0.076	450
reference	conservative	2,947	4.88			

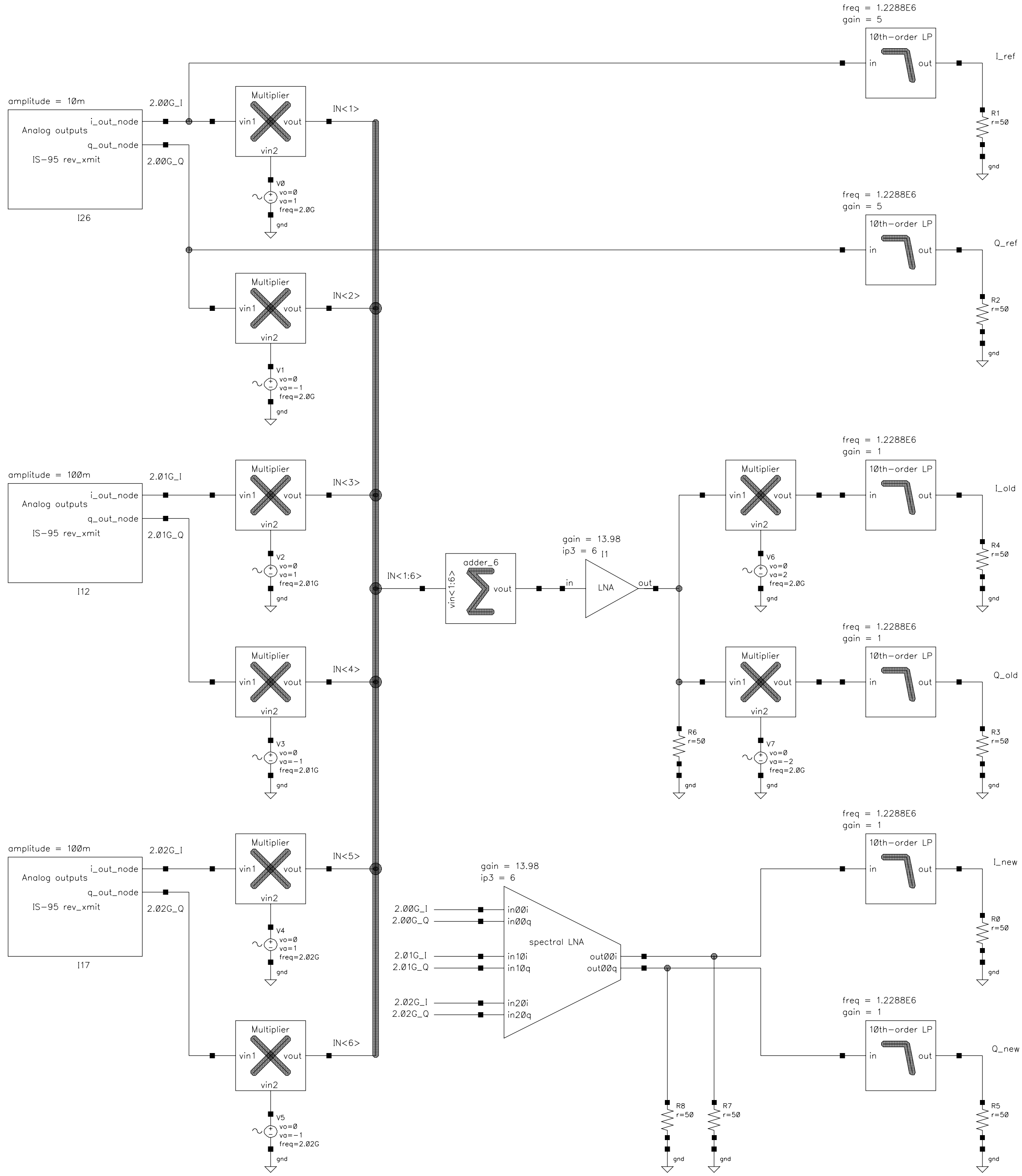

Fig. 4. Schematic of "master" benchmark circuit, from which sections are removed to run timing simulations.

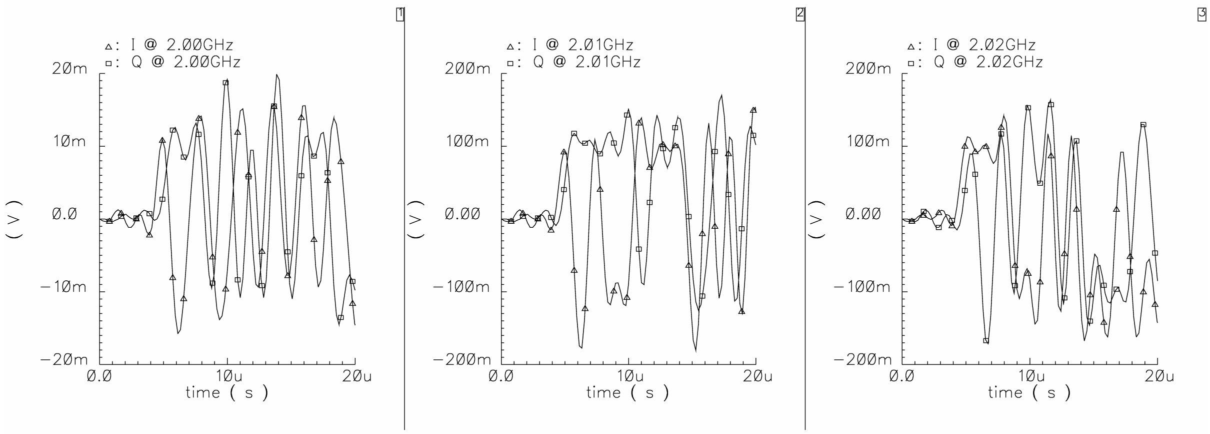

Fig. 5. Input modulation $I(t)$ and $Q(t)$, from left to right, the desired signal and two interference signals.

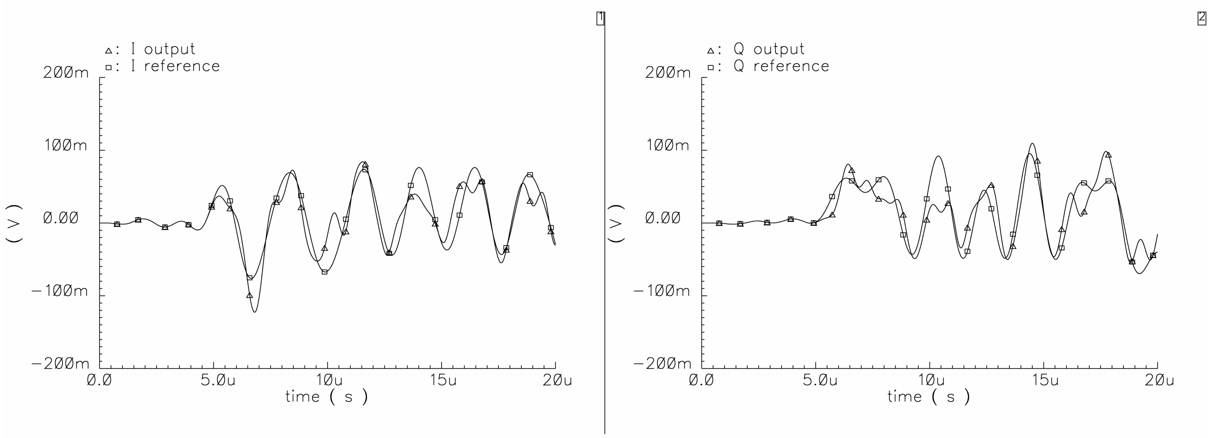

Fig. 6. Output modulation $I(t)$ and $Q(t)$, comparing ideal and distorted output signals.

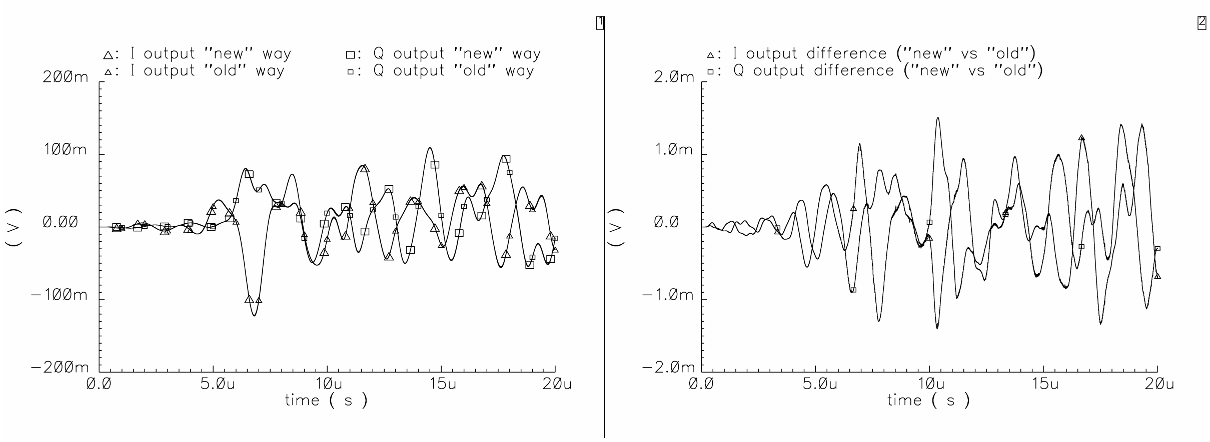

Fig. 7. Output modulation $I(t)$ and $Q(t)$, comparing "old" passband model and "new" spectral model results.