Variational Compact Modeling and Simulation for Linear Dynamic Systems *

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ABSTRACT
In this paper, we propose a new statistical model order reduction technique called SSMOR method, that is suitable for considering both intra-die and inter-die process variations. The SSMOR generates order reduced variational models from the original variational circuits. The reduced model can be used for fast statistical performance analysis of interconnect circuits with variational power sources. The SSMOR uses statistical spectrum method to compute the variational moments and Monte Carlo sampling method via modified Krylov subspace reduction method to generate the variational reduced models. Experimental results show that explicit moment matching is not suitable for variational analysis and Krylov subspace projection method is more reliable. The proposed method can deliver about 100X speedup over the pure Monte Carlo based projection-based reduction method with less than 1% of errors for both means and variances in statistical transient analysis.

1. INTRODUCTION
The process-induced variability has huge impacts on the circuit performance in the sub-100nm VLSI technologies [11, 10]. Process variational impacts on the interconnect circuit performance have to be assessed in the various VLSI design steps to ensure robust circuit design.

Statistical modeling of RLC interconnects, which are typically treated as linear time-invariant (LTI) dynamic systems, has been studied intensively in the past and many research works have been reported so far [2, 4, 7, 6, 9, 13]. Fundamentally, the most common approach to statistical modeling and simulation is Monte Carlo based sampling method, which is the most flexible and trusted method. However, its high computing costs render its applications limited to very small circuits.

Statistical modeling methods of interconnects based on the extracted parameters/variables were proposed in [7, 2, 6]. The idea is to treat the variational variables as the global variables (parameters) of the circuits. The original circuits then can be represented by matrix polynomial forms in terms of those variables. Thus, the traditional model order reduction methods are applied to the coefficient matrices of the polynomials. Those methods are more suitable for the inter-die variations as they can be treated as the global variables. Interval-valued statistical modeling and model order reduction methods have been proposed recently [9, 8]. The idea is to approximate the variations as a finite interval and uses the interval arithmetic to generate the order reduced models in terms of variational poles/residues and order reduced circuit matrices with interval valued parameters. Interval methods in general suffer the over-pessimism problem in spite of the recent improvement by using affine interval arithmetic. Also, the errors are accumulated with the arithmetic operations. Therefore, in [8], it was applied only to tree-like circuits, where solving the circuits can be done with very few numerical operations by topology tracing.

Another approach to the statistical modeling and simulation of interconnect circuits is by means of statistical spectrum analysis [5, 13, 4], where statistical variations are presented by orthogonal polynomials. One only needs to solve for the coefficients of the polynomials deterministically in order to compute the variations of the responses or performance metrics. The major benefit of this method is its compatibility with current transient simulation framework: it requires to solve for some coefficients of the orthogonal polynomials, which can be done using normal transient simulations of the original circuits with deterministic inputs, to compute variances of node responses. Ghanta [4] applied the statistical spectrum method to compute the time delays based on the frequency domain moment methods.

In this paper, we propose a new statistical spectrum based method, called Statistical Spectrum Model Order Reduction (SSMOR), to generate the order reduced variational models, which in turn can be used to compute the variational responses and performance matrices with given variational inputs. The variational models can be applied to fast statistical simulations of many interconnect circuits under various variations (both inter-die and intra-die). Our contribution is a new statistical model order reduction technique, which consists of the statistical spectrum method, the Krylov subspace based model order reduction technique, and Monte Carlo sampling method to generate order reduced variational models. The SSMOR follows a similar reduction flow proposed previously [8]. However, SSMOR uses the statistical spectrum method to compute the variational moments, which will not suffer the problems of over-pessimism and the operation-dependent accuracy loss of the interval-valued method. After variational moments are generated, a Monte Carlo sampling method is applied by using a modified Krylov subspace reduction method to generate the variational order reduced models. Since Monte Carlo only operates on the order reduced space (namely, a few moments), its high computing costs diminish.

The rest of this paper is organized as follows: Section 2 presents statistical modeling problem we try to solve. Section 3 reviews the orthogonal polynomial chaos based stochastic simulation methods. Section 4 presents our new statistical model order reduction method. Section 5 presents the experimental results and Section 6 concludes this paper.

2. PROBLEM FORMATION
Considering the following state equation for a given RLC interconnect circuit using Modified Nodal Analysis (MNA) formulation:

$$Gv(t) + C\frac{dv(t)}{dt} = Bu(t) \tag{1}$$

where $G \in R^{n \times n}$ is the conductance matrix, $C \in R^{n \times n}$ the matrix resulting from storage elements. $v(t)$ is the vector of time-varying node voltages and branch currents of voltage sources. $u(t)$ is the vector of independent power sources, and B is the input selector matrix.

The G and C matrices and input currents $u(t)$ depend on the circuit parameters, such as metal wire width, length, thickness on interconnects, and transistor parameters, like channel length, width, gate oxide thickness, etc. In this paper, all the circuit parameter variations are treated as uncorrelated Gaussian random variables as in [5] to model the intra-die variations. The spatial correlations can be removed by using a new set of independent random variables via orthogonal transformation methods, such as principal component analysis or Karhunen-Loeve expansion [3].

In this paper, we assume there are a number of independent (uncorrelated) transformed ortho-normal random Gaussian variables $\xi_i(\theta), i = 1, ..., n$, which actually model the channel length and the device threshold voltage variations. Let Θ denote the process sampling space. Let $\theta \in \Theta$, $\xi_i : \theta \to R$ denote a normalized Gaussian variable and $\xi(\theta) = [\xi_1(\theta), ..., \xi_n(\theta)]$ is a vector of n independent Gaussian variables. So the matrices G and C are functions of ξ, $G(\xi)$ and $C(\xi)$. The (1) becomes

$$G(\xi)v(t) + C(\xi)\frac{dv(t)}{dt} = Bu(t) \tag{2}$$

Note that input $u(t)$ is also subject to variations. But here we focus on the variations of the interconnects for the sake of modeling. The problem we try to solve is to produce a variational order reduced system in terms of $\hat{G} \in R^{k \times k}$ and $\hat{C} \in R^{k \times k}$, where $k << n$,

$$\hat{G}v(t) + \hat{C}\frac{dv(t)}{dt} = \hat{B}u(t) \tag{3}$$

where $\hat{G}$ and $\hat{C}$ have variational matrix elements, which can be treated as uncorrelated. The reduced models can be used with Monte Carlo method to compute the variational responses of interconnects like power grid and clock networks, due to variational input sources.

The reduced system can also be represented in terms of variational pole/residue forms, when transfer functions are computed for fast transient waveform computation by using the recursive convolution method.

3. STATISTICAL SPECTRUM ANALYSIS
In this section, we briefly review the statistical spectrum or orthogonal polynomial chaos (PC) based stochastic simulation methods.

3.1 Concept of Hermite Polynomial Chaos
In the following, a random variable $\xi(\theta)$ is expressed as a function of θ, which is the random event. Hermite PC utilizes a series of orthogonal polynomials (with respect to the Gaussian distribution) to facilitate stochastic analysis [14]. These polynomials are used as orthogonal basis to decompose a random process in a similar way that sine and cosine functions are used to decompose a periodic signal in a Fourier series expansion.

For a random variable $v(t,\xi)$ with limited variance, where $\xi = [\xi_1, \xi_2, ...\xi_n]$ is a vector of zero mean ortho-normal Gaussian random variables. The random variable can be approximated by truncated Hermite PC expansion as follows: [3]:

$$v(t,\xi) = \sum_{k=0}^{P} a_k H_k^n(\xi) \tag{4}$$

where n is the number of independent random variables, $H_k^n(\xi)$ is n-dimensional Hermite polynomials and a_k are the deterministic coefficients. The number of terms P is given

$$P = \sum_{k=0}^{p} \frac{(n-1+k)!}{k!(n-1)!} \tag{5}$$

where p is the order of the Hermite PC. If only one random variable is considered, the one-dimensional Hermite polynomials are

expressed as follows:

$$H_0^1(\xi) = 1, H_1^1(\xi) = \xi, H_2^1(\xi) = \xi^2 - 1, H_3^1(\xi) = \xi^3 - 3\xi, ... \tag{6}$$

Hermite polynomials are orthogonal with respect to Gaussian weighted expectation (the superscript n is dropped for simple notation):

$$< H_i(\xi), H_j(\xi) >=< H_i^2(\xi) > \delta_{ij} \tag{7}$$

where δ_{ij} is the Kronecker delta and $< *, * >$ denotes an inner product defined as follow:

$$< f(\xi), g(\xi) >= \frac{1}{\sqrt{(2\pi)^n}} \int f(\xi)g(\xi)e^{-\frac{1}{2}\xi^T\xi}d\xi \tag{8}$$

Like Fourier series, the coefficient a_k can be found by a projection operation onto the Hermite PC basis:

$$a_k(t) = \frac{< v(t,\xi), H_k(\xi) >}{< H_k^2(\xi) >}, \forall k \in \{0, ..., P\}. \tag{9}$$

3.2 Simulation Approach Based on Hermite PCs
In case that $v(t,\xi)$ is unknown random variable vector (with unknown distributions) like node voltages in (1), then the coefficients can be computed by using Galerkin method, which states that the best approximation of $v(t,\xi)$ is obtained when the error, $\Delta(t,\xi)$, is defined as

$$\Delta(t,\xi) = Gv(t) + C\frac{dv(t)}{dt} - I(t, \xi(\theta)) \tag{10}$$

is orthogonal to the approximation. That is

$$< \Delta(t,\xi), H_k(\xi) >= 0, i = 0, 1, ..., P \tag{11}$$

In this way, we transform the stochastic analysis process into a deterministic form, where we only need to compute the coefficients of its Hermite PC. Once we obtain those coefficients, the mean and variance of the random variables can be easily computed as shown later in the section.

For illustration purpose, considering one Gaussian variable $\xi = [\xi_1]$, we assume that the node voltage response can be written as a second order ($p = 2$) Hermite PC:

$$v(t,\xi) = v_0(t) + v_1(t)\xi_1 + v_2(t)(\xi_1^2 - 1) \tag{12}$$

Assuming that the input leakage current sources can also be represented by a second Hermite PC:

$$I(t,\xi) = I_0(t) + I_1(t)\xi_1 + I_2(t)(\xi_1^2 - 1) \tag{13}$$

By applying the Galerkin equation (11) and note that orthogonal property of the various order of Hermite PCs, we end up with the following three equations: (where $i = 0, 1, 2$)

$$Gv_i(t) + C\frac{dv_i(t)}{dt} = I_i(t) \tag{14}$$

For two independent Gaussian variables, we have

$$\begin{aligned} v(t,\xi) =\ & v_0(t) + v_1(t)\xi_1 + v_2(t)\xi_2 + v_3(t)(\xi_1^2 - 1) + \\ & v_4(t)(\xi_2^2 - 1) + v_5(t)(\xi_1\xi_2) \end{aligned} \tag{15}$$

Assuming that we have a similar second order Hermite PC for input leakage current $I(t,\xi)$,

$$\begin{aligned} I(t,\xi) =\ & I_0(t) + I_1(t)\xi_1 + I_2(t)\xi_2 + I_3(t)(\xi_1^2 - 1) + \\ & I_4(t)(\xi_2^2 - 1) + I_5(t)(\xi_1\xi_2) \end{aligned} \tag{16}$$

The (14) is still valid but with $i = 0, ..., 5$. For more (more than two) Gaussian variables, we can obtained the similar results (with more coefficients of Hermite PCs to solve by using (14).

Once we obtain the Hermite PC of $v(t,\xi)$, we can obtain the mean and variance of $v(t,\xi)$ trivially as (one Gaussian variable case):

$$
\begin{aligned}
E(v(t,\xi)) &= v_0(t) \\
Var(v(t,\xi)) &= v_1^2(t)Var(\xi_1) + v_2^2(t)Var(\xi_1^2 - 1) \\
&= v_1^2(t) + 2v_2^2(t) \quad (17)
\end{aligned}
$$

In the following section, we will show how to apply the statistical spectrum method to compute the variational circuit moments, which in turn are used to generate the variational reduced models via Krylov subspace reduction methods.

4. STATISTICAL SPECTRUM MODEL ORDER REDUCTION (SSMOR)

In this section, we first present our modified Krylov subspace model order reduction framework, which is suitable for variational modeling. Then we present the new variational moment computation method.

4.1 Modified Krylov Subspace Model Order Reduction

Krylov subspace based MOR method is to project the original circuit states into the dimension-reduced Krylov subspace of the circuit states. The Krylov subspace essentially is spanned by the dominant moment vectors of circuit transfer function. For a state space equation of an RLC circuit in (2), Krylov subspace is defined as

$$
K_q(A, \mathbf{b}) = \text{span}\{\mathbf{b}, A\mathbf{b}, A^2\mathbf{b}, ..., A^q\mathbf{b}\}, \quad (18)
$$

where $A = G^{-1}C$ and $\mathbf{b} = G^{-1}B$ and q is some given positive integer. Note that $A^i\mathbf{b}$ is the i block moment defined as

$$
\mathbf{m}_i = A^i\mathbf{b} = (-G^{-1}C)^i G^{-1}B, \quad (19)
$$

of the circuit state transfer function defined as $H(s) = (G + Cs)^{-1}B$. The block moment $\mathbf{m}_i$ can be directly computed in a recursive way

$$
\begin{aligned}
\mathbf{m}_0 &= G^{-1}B; \\
\mathbf{m}_1 &= -G^{-1}C\mathbf{m}_0; \\
&\cdots \\
\mathbf{m}_i &= -G^{-1}C\mathbf{m}_{i-1}; \text{ for } i > 0, \quad (20)
\end{aligned}
$$

One way to build the reduced model is by means of Pade approximation, which computes the poles/resides of the transfer functions by using the moment information directly, as shown in the classic AWE method [12]. However, this explicit moment matching method is not numerical stable for computing higher order models. Also, we will show in the experimental section that AWE-like method is not suitable for variational analysis, as the method is very sensitive to the variational changes. In other words, a small parameter change can lead to dramatic pole and residue value change, which makes the reduced models useless.

In our approach, we propose a modified Krylov subspace projection based MOR method to generate the reduced models. Specifically, we first define the moment matrix M as

$$
M = [\mathbf{m}_0, \mathbf{m}_1, ..., \mathbf{m}_{q-1}] \quad (21)
$$

The standard Krylov subspace projection method is to orthonormalize the vectors in M in order to generate a projection matrix V with the same dimension. Numerical methods like Arnoldi and Lanczos methods are typically used for the orthonormalization process, where the moment vectors are orthonormalized immediately after generation against all the previously-generated moment vectors.

Such orthonormalization process, however, is not suitable for our variational modeling process, as it is difficult to pass the variational information through the orthonormalization process using the aforementioned statistical spectrum method. Instead, we only

compute all variational moments first by statistical spectrum method. After all the block moments and its variations are computed, we switch to the Monte Carlo sampling method to generate the variational reduced models. In each sampling run, we orthonormalize moment vectors in M using Gram-Schmidt or modified Gram-Schmidt orthonormalization algorithms to compute projection matrix V. Once the projection matrix V is obtained, the original circuit matrix G and C are reduced to dimension-reduced matrices by *congruence transformation*:

$$
\hat{G} = V^T GV; \hat{C} = V^T CV; \hat{B} = V^T B \quad (22)
$$

Due to the nature of *congruence transformation*, the reduction process guarantees the passivity of all the reduced models. To compute the poles and residues, we can further perform eigen-decomposition of $\hat{G}^{-1}\hat{C}$

$$
\hat{G}^{-1}\hat{C} = S\Lambda S^{-1}
$$

where $\Lambda = diag(\lambda_1, \lambda_2, ...\lambda_i)$, which are the reciprocals of the dominant poles.

To find the residues, we solve for w in $\hat{G}w = V^T B$. Then the residues are simply the multiplications of $S^T V^T B$ and $S^{-1}w$. Note that when we generate the variational reduced models using Monte-Carlo method, we need to consider the variations in both moments (thus the projection matrix) and the original G and C matrices in state equations. Although those variational in some sense are correlated, but we still treat them as uncorrelated during the Monte Carlo sampling to simplify the modeling process.

4.2 The New Statistical Model Order Reduction Flow

The proposed statistical model order reduction flow, SSMOR, is shown in the left hand side of Fig. 1. As comparison, we also show the pure Monte Carlo based MOR approach using the traditional Krylov subspace projection MOR method. In the new flow, we use statistical spectrum method to compute the variational moments first. After this, we switch to the Monte Carlo sampling method to generate the variational reduced models by using the modified Krylov subspace method. The samplings are done based on the computed means and variances for each moments using Gaussian distributions. Since the Monte Carlo method works on the reduced models, we gain significant speedup over the pure Monte Carlo MOR method.
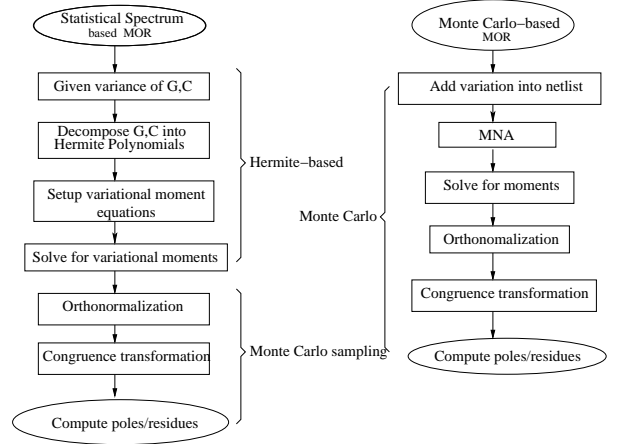

Figure 1: Flowchart of Statistical Spectrum and Monte Carlo Algorithms

4.3 Statistical Moment Computation

4.3.1 One Random Variable

In this case, the variational G and C matrices become

$$G = g_0 + g_1\xi \; ; C = c_0 + c_1\xi \qquad (23)$$

where, ξ is the random variable with Gaussian distribution with zero mean and standard deviation 1, i.e. $\xi \sim N(0,1)$. In addition, (g_0, g_1) and (c_0, c_1) denote the mean and variance of G and C, respectively.

The Hermite polynomial expansion utilizes a series of orthogonal polynomials with respect to Gaussian distribution. These polynomials are treated as an orthogonal basis to decompose a random process. We choose second-order Hermite polynomial to represent the statistical process here, as Hermite polynomial has exponential error convergence [3]. In some cases, first order Hermite polynomial is sufficiently accurate, which can significantly simplify the moment computation process, as it will be shown in the case of multiple random variables later.

As we know, the basis of Hermite polynomial at the order of two is defined as $[1, \xi, \xi^2 - 1]^T$. Thus, the zero moment of a given RC circuit can be decomposed as:

$$m_0 = [a_{m0}, a_{m1}, a_{m2}][1, \xi, \xi^2 - 1]^T \qquad (24)$$

Similarly, the (2q)th moment and (2q-1)th moment can be decomposed as

$$m_{2q} = [a_0, a_1, a_2][1, \xi, \xi^2 - 1]^T \; ; m_{2q-1} = [b_0, b_1, b_2][1, \xi, \xi^2 - 1]^T$$

where, $[a_{m0}, a_{m1}, a_{m2}]$, $[a_0, a_1, a_2]$, and $[b_0, b_1, b_2]$ are coefficients with respect to the Hermite polynomial basis.

Applying the principals of orthogonality, or so-called Galerkin method, in Hermite polynomials, we have

$$< \triangle p, 1 >= 0; < \triangle p, \xi >= 0; < \triangle p, \xi^2 - 1 >= 0$$

where $<,>$ denotes the inner product, and $\triangle p$ is the truncation error caused by Hermite polynomial expansion. As a result, the zero moment can be solved with the following equation:

$$\begin{bmatrix} g_0 & g_1 & 0 \\ g_1 & g_0 & 2g_1 \\ 0 & 4g_1 & 4g_0 \end{bmatrix} \begin{bmatrix} a_{m0} \\ a_{m1} \\ a_{m2} \end{bmatrix} - \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} = 0 \qquad (25)$$

Therefore, the $(2q)$th moment can be derived from $(2q-1)$th moment in a recursive way as follows:

$$\begin{bmatrix} g_0 & g_1 & 0 \\ g_1 & g_0 & 2g_1 \\ 0 & 2g_1 & 2g_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} c_0 & c_1 & 0 \\ c_1 & c_0 & 2c_1 \\ 0 & 2c_1 & 2c_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = 0$$

$$(26)$$

where, $[a_{m0}, a_{m1}, a_{m2}]$, $[a_0, a_1, a_2]$, and $[b_0, b_1, b_2]$ are coefficients with respect to the Hermite polynomial basis.

4.3.2 Multiple Random Variables

Let's consider n random variables. In this case, we only use first order Hermite expansion. This is also a valid assumption practically as first-order Hermite polynomial leads directly to Gaussian distributions. Many interconnect timing performances can be assumed to be Gaussian given a Gaussian variations on the RLC elements [1, 4]. The variational G and C matrices now become

$$G = g_0 + \sum_{i=1}^{n} g_i\xi_i \; ; C = c_0 + \sum_{i=1}^{n} c_i\xi_i$$

For n random variables, the basis of Hermite polynomials with one order is known as $[1, \xi_1, \xi_2, ..., \xi_n]$. Thus,

$$m_0 = a_{m0} + \sum_{i=1}^{n} a_{mi}\xi_i \; ; m_{2q} = a_0 + \sum_{i=1}^{n} a_i\xi_i \; ; m_{2q-1} = b_0 + \sum_{i=1}^{n} b_i\xi_i$$

where, $[a_{m0}, a_{m1}, ..., a_{mn}]$, $[a_0, a_1, ..., a_n]$, and $[b_0, b_1, ..., b_n]$ are coefficients with respect to the Hermite polynomial basis. Applying

the principals of orthogonality and equalities of Gaussian distributions, the zero moment can be computed with the following equation:

$$\begin{bmatrix} g_0 & g_1 & g_2 & \cdots & g_i & \cdots & g_n \\ g_1 & g_0 & 0 & \cdots & 0 & \cdots & 0 \\ g_2 & 0 & g_0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_i & 0 & 0 & \cdots & g_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_n & 0 & 0 & \cdots & \cdots & \cdots & g_0 \end{bmatrix} \begin{bmatrix} a_{m0} \\ a_{m1} \\ a_{m2} \\ \vdots \\ a_{mi} \\ \vdots \\ a_{mn} \end{bmatrix} - \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0$$

$$(27)$$

Once the zero moment is computed, the $(2q)$th moment can be evaluated from $(2q-1)$th moment recursively with the following equation:

$$\begin{bmatrix} g_0 & g_1 & g_2 & \cdots & g_i & \cdots & g_n \\ g_1 & g_0 & 0 & \cdots & 0 & \cdots & 0 \\ g_2 & 0 & g_0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_i & 0 & 0 & \cdots & g_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_n & 0 & 0 & \cdots & \cdots & \cdots & g_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix} +$$

$$\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_i & \cdots & c_n \\ c_1 & c_0 & 0 & \cdots & 0 & \cdots & 0 \\ c_2 & 0 & c_0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_i & 0 & 0 & \cdots & c_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_n & 0 & 0 & \cdots & \cdots & \cdots & c_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix} = 0 \qquad (28)$$

Once the moments and their variations are computed by statistical spectrum method, we proceed to compute the variational models via sampling based Monte Carlo methods by using modified Krylov subspace projection methods, as mentioned in the earlier part of this section.

5. EXPERIMENTAL RESULTS

This section describes the simulation results of circuits with variations in G and C in linear dynamic systems. The proposed method has been implemented in Matlab and partially in Perl. All the experimental results are carried out in Linux system with dual Xeon CPU's with 3.06 GHz and 1 GB of memory.

We first start with one variation with given standard variation, which affects both G and C in linear system. For pure Monte Carlo based model order reduction method, we perform the modified Krylov subspace method on the variational G and C matrices. Specifically, for each sampled linear dynamic circuit, we first find the first qth order moments, $q = 10$, of the system in our experiment in a recursive way. Once the moments are solved, the next step is to find the corresponding reduced circuit matrices, $\hat{G}$, $\hat{C}$ and $\hat{B}$. Finally, we find the pole and residues by eigen-decomposition. At least five poles are evaluated in our experiments.

In the SSMOR method, we compute variational moments by using the statistical spectrum method. After variational moments are computed, we switch to Monte Carlo to compute the variational poles and residues. To compute the poles and residues, we use two approaches: one is the explicit moment matching method like AWE, and the other is modified Krylov subspace projection based method.

We select a small RLC network with about 33 nodes and some variational current sources to test the proposed method. The small size

of the circuit allows Mente Carlo simulations to finish within reasonable time. The variance for the *R* and *C* is about 0.005. The variances for current sources are set to 0.01. We tested larger circuits in Table 2 to study the scalability of the proposed method over the pure Monte Carlo method.

Fig. 2 shows the comparison between SSMOR approach and Monte Carlo simulation with one random variable in terms of pole variations. Given the same circuit for both methods, the experiment is
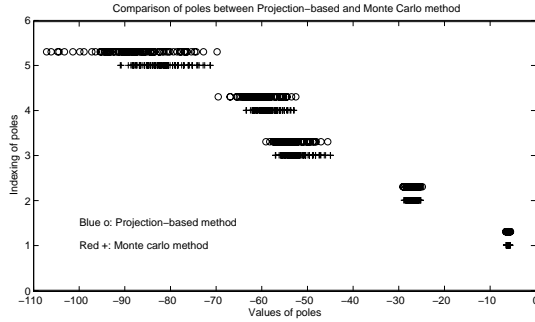
Figure 2: Comparison of poles between SSMOR and Monte Carlo methods (one random variable).

repeated for 2000 times in Monte Carlo method, such that there are 2000 sets of moments for each method. The number of samples is sufficient to guarantee a 99% confidence level with 1% to 2% inaccuracy. The values of poles are derived from those 2000 sets of moments using the SSMOR method, the pure Monte Carlo MOR method. The values of poles are shown in *x*-axis with five pole indices shown in *y*-axis. We can see that the SSMOR method agrees pretty well with the pure Monte Carlo MOR method using Krylov subspace methods.

We also report the computed pole variations by using explicit moment matching methods like AWE. One observation is that the resulting pole variations have large discrepancy with the projection-based method as shown in Fig. 3 even for one random variable. This further confirms that explicit moment matching methods are very sensitive to numerical noises and are not suitable for variational modeling and analysis. The pole values derived by AWE algorithm in the figure may show positive values in real parts of the pole values (complex numbers) due to variational changes. However, all the poles are negative numbers, thus the system is stable, by using our SSMOR method.
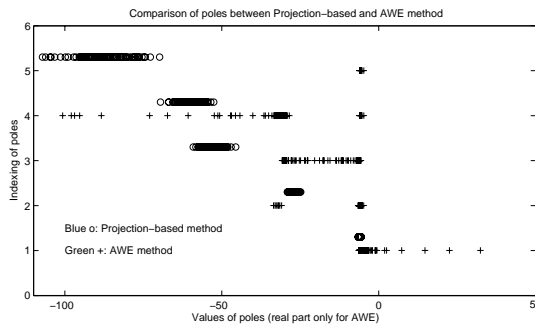
Figure 3: Comparison of poles between projection-based and AWE method (one random variable) in SSMOR.

Similarly, we compare the pole variations between the new SSMOR method and the pure Monte Carlo MOR method for two random variables. Two random variables are assumed to be un-

correlated and affect both G and C at the same time in the system. The results are shown in Fig. 4. The real part of pole values are shown in *x*-axis for five poles. Again, the result from SSMOR method matches well with the pure Monte Carlo method. We also
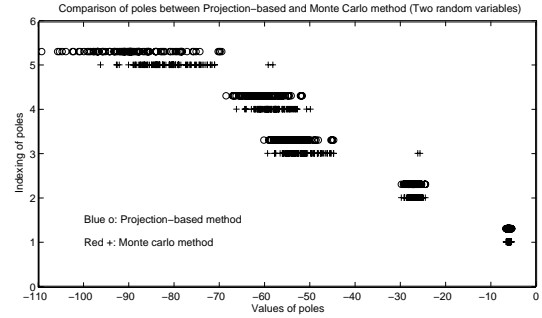
Figure 4: Comparison of poles between SSMOR and Monte Carlo methods (two random variables)

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gives results by using AWE compared with projection based MOR in SSMOR method in Fig. 5. Again, the pole variations in AWE are significantly different from the projection-based method. Finally,
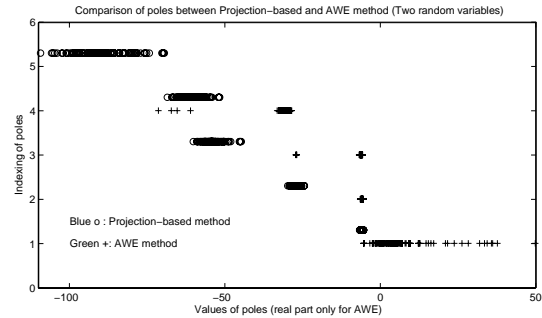

Figure 5: Comparison of poles between projection-based and AWE method (two random variables) in SSMOR.

we use the reduced variational models to compute the transient responses with deterministic and variational power source inputs. The variational inputs are piecewise linear current sources.

Given a deterministic piecewise linear input, the comparison between SSMOR and the Monte Carlo MOR method with 2000 samplings is shown in Fig. 6. For the variational models, we use recursive convolution method to compute the transient responses after variational poles and residues are computed. The two waveforms are very similar.

In the case of variational stimulus, the comparison between SSMOR method and Monte Carlo MOR method is shown in Fig. 7. As it can be shown in both figures, the responses from our SSMOR method are almost identical to the ones using Monte Carlo method.

To measure the error of SSMOR method against Monte Carlo method, we measure the transient waveforms at three different time instances at a selected node as shown in the first column of Table 1. The Table 1 shows the percentage of errors for the three time instances over 2000 samples. It can be seen that the transient errors between two methods are less than 1% for both means and variances.
In consideration of the runtime speed between SSMOR and Monte Carlo MOR, the result of the speedup is shown in Table 2. Please note that the measurement of the speedup is based on the algorithms for SSMOR and Monte Carlo MOR. The benchmark includes the computation of poles and residues. However, it does not include
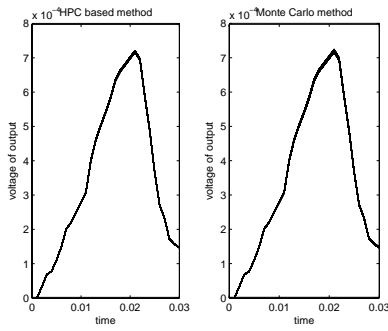
Figure 6: Comparison of PWL response between SSMOR reduced model and Monte Carlo method with deterministic stimulus (two random variables).
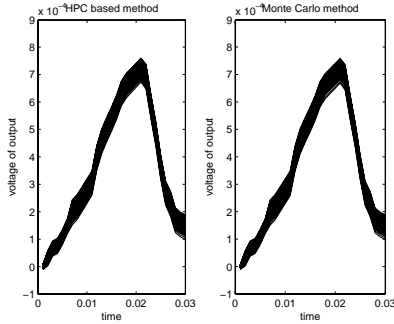
Figure 7: Comparison of PWL response between SSMOR reduced model and Monte Carlo method with stochastic stimulus (two random variables).

the time for transient analysis. The SSMOR shows about 100X of speedup over pure Monte Carlo depending upon the size of the circuits.

6. CONCLUSION AND FUTURE WORK

We have proposed a new statistical model order reduction technique, named SSMOR, that is suitable for considering both intra-die and inter-die process variations. The SSMOR generates order reduced variational models from the original variational circuits. The reduced model can be used for fast statistical performance analysis of interconnect circuits with variational power sources. The SSMOR method combines the statistical spectrum analysis method, Monte Carlo sampling method, and modified Krylov subspace model order reduction technique, to generate the statistical reduced models. Experimental results show that explicit moment matching is not suitable for variational analysis and Krylov subspace projection method is more reliable. The proposed SSMOR method can deliver about 100X speedup over the pure Monte Carlo projection-based reduction method with less than 1% of errors in statistical transient analysis.

7. REFERENCES

Table 1: Voltage response comparison between SSMOR and Monte Carlo methods

Time instance (e-3) s	SSMOR		MC		% error	
	mean (e-5)	std (e-6)	mean (e-5)	std (e-6)	mean %	std %
3	6.852	9.518	6.835	9.487	0.25	0.327
5	11.11	9.489	11.078	9.454	0.253	0.369
20	70.27	15.56	70.1	15.45	0.232	0.732

Table 2: Runtime comparison between SSMOR and Monte Carlo method

	#node	SSMOR	MC	Speedup
Ckt1	33	1	37.03	37 times
Ckt2	553	1	162.16	162 times
Ckt3	1720	1	118.03	118 times

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